RELATIVITY AND COSMOLOGY I

Solutions to Problem Set 6 Fall 2023

1. The Expanding Universe

(a) This is a diagonal metric, so we know that $\Gamma^0_{ij} \propto \delta_{ij}$. Moreover, the only derivatives that do not vanish are the ones with respect to time, of the spatial components of the metric. This means that the only non vanishing Christoffel symbols are

$$\Gamma^0_{ij} = -\frac{1}{2g_{00}}\partial_0 g_{ij} = a\dot{a}\delta_{ij}\,, \qquad \Gamma^i_{0j} = \partial_0\left(\ln\sqrt{|g_{ij}|}\right) = \frac{\dot{a}}{a}\delta^i_j\,. \tag{1}$$

(b) Consider a null path in the x direction

$$0 = -\mathrm{d}t^2 + a^2(t)\mathrm{d}x^2\,. \tag{2}$$

From this, we deduce that

$$\frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda}\,. \tag{3}$$

It might seem that we have simply used $\mathrm{d}t^2$ and $\mathrm{d}x^2$ as factors to be divided and interpreted as differentials. Most of the times, that works. But there is a more formal way to understand what we have done: given a curve $x^\mu(\lambda)$, its infinitesimal length is given by

$$\sqrt{g_{\mu\nu}(x(\lambda))\frac{dx^\mu(\lambda)}{d\lambda}\frac{dx^\nu(\lambda)}{d\lambda}}d\lambda = \sqrt{-\left(\frac{dt}{d\lambda}\right)^2 + a^2(t)\left(\frac{dx}{d\lambda}\right)^2}d\lambda, \tag{4}$$

and a null path has by definition length, and henceforth infinitesimal length, zero, from which the result follows (up to a sign corresponding to the lightray motion in positive and negative directions).

We then found that null paths satisfy

$$\frac{dx}{d\lambda} = \frac{1}{a}\frac{dt}{d\lambda}\,, \tag{5}$$

where we chose the plus sign to consider paths that move in the positive x direction. To find the momentum of a photon, we also need to impose that $x^\mu(\lambda)$ is a geodesic, on top of being a null path. Since we are interested in an observer at rest, $U^\mu_{\text{obs}} = (1,0,0,0)$, effectively only $\frac{dt}{d\lambda}$ will be relevant to compute the energy of the photon. We study its component in the geodesic equation

$$\begin{aligned}
\frac{d^2t}{d\lambda^2} + \Gamma^0_{11}\left(\frac{dx}{d\lambda}\right)^2 &= 0 \\
\frac{d^2t}{d\lambda^2} + a\dot{a}\left(\frac{dx}{d\lambda}\right)^2 &= 0 \\
\frac{d^2t}{d\lambda^2} + \frac{\dot{a}}{a}\left(\frac{dt}{d\lambda}\right)^2 &= 0\,.
\end{aligned} \tag{6}$$

where in the last line we used (5). Defining $p^0(t) = \frac{dt}{d\lambda}$, the differential equation becomes

$$\begin{aligned} \frac{dp^0}{d\lambda} + \frac{\dot{a}}{a}(p^0)^2 = 0 \\ p^0\dot{p}^0 &= -\frac{\dot{a}}{a}(p^0)^2 \\ \frac{\dot{p}^0}{p^0} &= -\frac{\dot{a}}{a}\,. \end{aligned} \tag{7}$$

This equation is solved by

$$p^0 = \frac{E_0}{a}\,, \tag{8}$$

where E_0 is the value of p^0 when $a = 1$. The energy of the photon for an observer at rest is thus

$$E = -p_\mu U^\mu = -g_{00}\frac{E_0}{a} = \frac{E_0}{a}\,. \tag{9}$$

We interpret this result as the fact that the energy of the photon as measured by an observer at rest is diminished by the expansion of the universe: this is called the **cosmological redshift**.

(c) In the fluid's rest frame, the non-vanishing components of the stress energy tensor are

$$T^{00} = \rho\,, \qquad T^{ij} = \frac{P}{a^2}\delta_{ij}\,. \tag{10}$$

The conservation equation explicitly reads

$$\nabla_\mu T^{\mu\nu} = \partial_\mu T^{\mu\nu} + \Gamma^\mu_{\mu\sigma}T^{\sigma\nu} + \Gamma^\nu_{\mu\sigma}T^{\mu\sigma} = 0\,. \tag{11}$$

Choosing $\nu = 0$ we get the **continuity equation**

$$\begin{aligned} \partial_0 T^{00} + \Gamma^\mu_{\mu 0}T^{00} + \Gamma^0_{\mu\sigma}T^{\mu\sigma} &= 0 \\ \dot{\rho} + 3\frac{\dot{a}}{a}\rho + 3a\dot{a}\frac{P}{a^2} &= 0 \\ \dot{\rho} &= -3\frac{\dot{a}}{a}(\rho + P)\,. \end{aligned} \tag{12}$$

Choosing instead $\nu = i$ we obtain

$$\begin{aligned} \partial_i T^{ii} + \Gamma^\mu_{\mu i}T^{ii} + \Gamma^i_{\mu\sigma}T^{\mu\sigma} &= 0 \\ \frac{1}{a^2}\partial_i P &= 0\,, \end{aligned} \tag{13}$$

as we wanted to show.

(d) Using $P = \omega\rho$ in the continuity equation we get

$$\begin{aligned} \dot{\rho} &= -3\frac{\dot{a}}{a}\rho(1+\omega) \\ \frac{d\rho}{\rho} &= -3\frac{da}{a}(1+\omega) \\ \rho &= \rho_0 a^{-3(1+\omega)}\,. \end{aligned} \tag{14}$$

The various cases that we are given in the Problem set represent the different eras of evolution of our universe in which different kinds of fluids were dominating the total energy density and pressure of the universe. The condition $\omega = 0$ represents pressure-less matter, which dilutes as

$$\rho_m = \rho_{m,0} a^{-3} \,. \tag{15}$$

This makes sense, as the expansion of the universe increases the volume and thus reduces the density of matter. The condition $\omega = \frac{1}{3}$ represents radiation, which dilutes as

$$\rho_r = \rho_{r,0} a^{-4} \,. \tag{16}$$

As we saw in the first part of this exercise, the energy of photons is diluted by the cosmological redshift, so that on top of the cubic dilution of density due to the expansion of space, we have an extra power of a given by the stretching of the photons' wavelengths. Finally, the condition $\omega = -1$ represents dark energy, which has negative pressure. Its energy density goes as

$$\rho_\Lambda = \rho_{\Lambda,0} \,. \tag{17}$$

Can you deduce the chronological order of these different eras of matter domination, radiation domination and dark energy domination?

2. Geodesic Deviation

(a) The commutator between T and S vanishes because they are two different vectors of a basis. Explicitly,

$$\begin{aligned}[T,S]^\nu &= T^\mu \partial_\mu S^\nu - S^\mu \partial_\mu T^\nu \\ &= \frac{\partial x^\mu}{\partial t} \partial_\mu \frac{\partial x^\nu}{\partial s} - \frac{\partial x^\mu}{\partial s} \partial_\mu \frac{\partial x^\nu}{\partial t} \\ &= \frac{\partial^2 x^\nu}{\partial t \partial s} - \frac{\partial^2 x^\nu}{\partial s \partial t} = 0\end{aligned} \tag{18}$$

as expected. It will then be useful to use

$$[T,S]^\nu \equiv T^\mu \partial_\mu S^\nu - S^\mu \partial_\mu T^\nu = T^\mu \nabla_\mu S^\nu - S^\mu \nabla_\mu T^\nu, \tag{19}$$

which follows from the symmetry of the Christoffel symbols in their lower indices. Indeed the additional terms are

$$T^\mu \Gamma^\nu_{\mu\rho} S^\rho - S^\mu \Gamma^\nu_{\mu\rho} T^\rho = T^\mu \Gamma^\nu_{\mu\rho} S^\rho - S^\rho \Gamma^\nu_{\rho\mu} T^\mu = 0 \tag{20}$$

"Geometric" solution This exercise is easier by using some differential geometric language (you already know the concepts, it's just a smarter notation in which some things take a simpler form). If you don't like it is fine, there is an alternate solution using brute force on components below. In class you saw the definition of the Riemann tensor

$$[\nabla_\mu, \nabla_\nu] T^\lambda = R^\lambda{}_{\sigma\mu\nu} T^\sigma. \tag{21}$$

However there is a more general definition that uses the covariant derivative along a vector X, $\nabla_X \equiv X^\mu \nabla_\mu$. It has the geometric meaning of computing the derivative of a tensor in the direction of the vector X [1]. The ∇_μ you know is just a special case when X is a basis vector. Now the more general definition of the Riemann tensor is

$$\left([\nabla_X, \nabla_Y] - \nabla_{[X,Y]}\right) T^\lambda = R^\lambda{}_{\sigma\mu\nu} X^\mu Y^\nu T^\sigma. \tag{22}$$

As a bonus exercise [2] you can show that this definition is equivalent to the one you already now (it's just a few lines). Now we can rewrite (19) as

$$\nabla_T S - \nabla_S T = 0. \tag{25}$$

Then we have

$$A = \nabla_T V = \nabla_T \nabla_T S = \nabla_T \nabla_S T = [\nabla_T, \nabla_S] T, \tag{26}$$

where in the last line we used $\nabla_T T = 0$ which is just the geodesic equation. Now this is just the definition of the Riemann tensor (22) for $X = T$ and $Y = S$ and $[S, T] = 0$. Therefore,

$$A^\mu = [\nabla_T, \nabla_S] T^\mu = R^\mu{}_{\nu\rho\sigma} T^\rho S^\sigma T^\nu, \tag{27}$$

as we wanted to show.

Brute force solution Let us be very pedantic and carry out each step

$$\begin{aligned}
A^\mu &= T^\rho \nabla_\rho V^\mu \\
&= T^\rho \nabla_\rho (T^\sigma \nabla_\sigma S^\mu) \\
&= T^\rho \nabla_\rho (S^\sigma \nabla_\sigma T^\mu) \\
&= (T^\rho \nabla_\rho S^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma \nabla_\rho \nabla_\sigma T^\mu \\
&= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma (\nabla_\rho \nabla_\sigma T^\mu - \nabla_\sigma \nabla_\rho T^\mu + \nabla_\sigma \nabla_\rho T^\mu) \\
&= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + T^\rho S^\sigma (R^\mu{}_{\nu\rho\sigma} T^\nu + \nabla_\sigma \nabla_\rho T^\mu) \\
&= (S^\rho \nabla_\rho T^\sigma)(\nabla_\sigma T^\mu) + R^\mu{}_{\nu\rho\sigma} T^\rho T^\nu S^\sigma + S^\sigma \nabla_\sigma (T^\rho \nabla_\rho T^\mu) - (S^\sigma \nabla_\sigma T^\rho) \nabla_\rho T^\mu \\
&= R^\mu{}_{\nu\rho\sigma} T^\rho T^\nu S^\sigma + S^\sigma \nabla_\sigma (T^\rho \nabla_\rho T^\mu) \\
&= R^\mu{}_{\nu\rho\sigma} T^\rho T^\nu S^\sigma \,,
\end{aligned} \tag{28}$$

proving what we wanted. The step from the second to the third line is using the vanishing of the commutator of the two vectors. The step from the fourth line to the fifth is introducing some terms that add up to zero in order to make the Riemann tensor appear. After that, we write $T^\rho S^\sigma \nabla_\sigma \nabla_\rho T^\mu = S^\sigma \nabla_\sigma (T^\rho \nabla_\rho T^\mu) -$

[1] Note that the notion of derivative is meaningless without specifying a direction, if you are surprised it is just that this fact was hidden to you when always taking derivatives along basis vectors.

[2] Here it is :

$$[\nabla_X, \nabla_Y] T^\lambda = X^\mu Y^\nu [\nabla_\mu, \nabla_\nu] T^\lambda + X^\mu (\nabla_\mu Y^\nu) \nabla_\nu T^\lambda - Y^\nu (\nabla_\nu X^\mu) \nabla_\mu T^\lambda, \tag{23}$$

where the second term comes from Leibniz's rule. It gets cancel by the covariant derivative along the commutator :

$$\nabla_{[X,Y]} T^\lambda = [X, Y]^\nu \nabla_\nu T^\lambda = (X^\mu (\nabla_\mu Y^\nu) - Y^\mu (\nabla_\mu X^\nu)) \nabla_\nu T^\lambda. \tag{24}$$

$(S^\sigma \nabla_\sigma T^\rho)\nabla_\rho T^\mu$ such that in the next line we get a neat cancelation. The final step is true because $T^\rho \nabla_\rho T^\mu = 0$ for a vector T that is defined to be tangent to a geodesic.

3. The Riemann Tensor

Following the hint, we will start from

$$R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} = 0\,, \tag{29}$$

and its permutations

$$R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} + R_{\sigma\rho\mu\nu} = 0\,, \tag{30}$$
$$R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} = 0\,, \tag{31}$$
$$R_{\nu\rho\sigma\mu} + R_{\nu\sigma\mu\rho} + R_{\nu\mu\rho\sigma} = 0\,. \tag{32}$$

We start by adding (29) to (32). We get

$$\begin{aligned} 0 &= R_{\rho\sigma\mu\nu} + R_{\rho\mu\nu\sigma} + R_{\rho\nu\sigma\mu} + R_{\nu\rho\sigma\mu} + R_{\nu\sigma\mu\rho} + R_{\nu\mu\rho\sigma} \\ &= R_{\rho\sigma\mu\nu} + R_{\mu\rho\sigma\nu} + R_{\rho\nu\sigma\mu} - R_{\rho\nu\sigma\mu} + R_{\sigma\nu\rho\mu} + R_{\nu\mu\rho\sigma} \\ &= R_{\rho\sigma\mu\nu} + R_{\mu\rho\sigma\nu} + R_{\sigma\nu\rho\mu} + R_{\nu\mu\rho\sigma}\,, \end{aligned} \tag{33}$$

where we used the Riemann's antisymmetry properties to cancel some terms and put other terms in a form that will be convenient for what is to come. We continue by adding (30) to (31). We get

$$\begin{aligned} 0 &= R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} + R_{\sigma\rho\mu\nu} + R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} + R_{\mu\sigma\nu\rho} \\ &= R_{\sigma\mu\nu\rho} + R_{\sigma\nu\rho\mu} - R_{\rho\sigma\mu\nu} + R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} - R_{\sigma\mu\nu\rho} \\ &= R_{\sigma\mu\nu\rho} - R_{\rho\sigma\mu\nu} + R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu}\,. \end{aligned} \tag{34}$$

We impose this to be equivalent to (33). We obtain

$$\begin{aligned} R_{\rho\sigma\mu\nu} + R_{\mu\rho\sigma\nu} + R_{\sigma\nu\rho\mu} + R_{\nu\mu\rho\sigma} &= R_{\sigma\mu\nu\rho} - R_{\rho\sigma\mu\nu} + R_{\mu\nu\rho\sigma} + R_{\mu\rho\sigma\nu} \\ R_{\rho\sigma\mu\nu} - R_{\mu\nu\rho\sigma} &= -R_{\rho\sigma\mu\nu} + R_{\mu\nu\rho\sigma} \\ 2R_{\rho\sigma\mu\nu} &= 2R_{\mu\nu\rho\sigma}\,, \end{aligned} \tag{35}$$

as we wanted to prove.

4. The Curvature of S^3

(b) The integral reads

$$I = \frac{1}{2}\int \left[\left(\frac{d\psi}{d\lambda}\right)^2 + \sin^2\psi\left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right)\right] d\lambda \tag{36}$$

Let us consider first the variation with respect to $\psi \to \psi + \delta\psi$. We are going to only keep terms up to $\delta\psi^2$

$$\begin{aligned}\delta_\psi I &= \frac{1}{2}\int \left[2\frac{d\psi}{d\lambda}\frac{d\delta\psi}{d\lambda} + 2\sin\psi\cos\psi\delta\psi\left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right)\right] d\lambda \\ &= \int \left[-\frac{d^2\psi}{d\lambda^2} + \sin\psi\cos\psi\left(\left(\frac{d\theta}{d\lambda}\right)^2 + \sin^2\theta\left(\frac{d\phi}{d\lambda}\right)^2\right)\right]\delta\psi d\lambda\,.\end{aligned} \tag{37}$$

We used that

$$\left(\frac{d\psi}{d\lambda} + \frac{d\delta\psi}{d\lambda}\right)^2 = \left(\frac{d\psi}{d\lambda}\right)^2 + 2\frac{d\psi}{d\lambda}\frac{d\delta\psi}{d\lambda} + O(\delta\psi^2)\,, \tag{38}$$

and integrated by parts the derivative acting on $\delta\psi$. The variation of this integral should vanish. Comparing with the geodesic equation, we read off some Christoffel symbols

$$\Gamma^\psi_{\theta\theta} = -\sin\psi\cos\psi\,, \qquad \Gamma^\psi_{\phi\phi} = -\sin\psi\cos\psi\sin^2\theta\,. \tag{39}$$

Then we vary with respect to θ and we get

$$\begin{aligned}\delta_\theta I &= \frac{1}{2}\int \left[\sin^2\psi\left(2\frac{d\theta}{d\lambda}\frac{d\delta\theta}{d\lambda} + 2\sin\theta\cos\theta\delta\theta\left(\frac{d\phi}{d\lambda}\right)^2\right)\right] d\lambda \\ &= \int \left[-2\sin\psi\cos\psi\frac{d\psi}{d\lambda}\frac{d\theta}{d\lambda} - \sin^2\psi\frac{d^2\theta}{d\lambda^2} + \sin^2\psi\sin\theta\cos\theta\left(\frac{d\phi}{d\lambda}\right)^2\right]\delta\theta d\lambda\,.\end{aligned} \tag{40}$$

We read off

$$\Gamma^\theta_{\psi\theta} = \cot\psi\,, \qquad \Gamma^\theta_{\phi\phi} = -\sin\theta\cos\theta\,. \tag{41}$$

Finally, we vary with respect to ϕ.

$$\begin{aligned}\delta_\phi I &= \int \sin^2\psi\sin^2\theta\frac{d\phi}{d\lambda}\frac{d\delta\phi}{d\lambda}d\lambda \\ &= \int \Big[-2\sin^2\psi\sin\theta\cos\theta\frac{d\theta}{d\lambda}\frac{d\phi}{d\lambda} - 2\sin\psi\cos\psi\sin^2\theta\frac{d\psi}{d\lambda}\frac{d\phi}{d\lambda} \\ &\qquad\qquad - \sin^2\theta\sin^2\psi\frac{d^2\phi}{d\lambda^2}\Big]d\lambda\delta\phi\,.\end{aligned} \tag{42}$$

We read off

$$\Gamma^\phi_{\theta\phi} = \cot\theta\,, \qquad \Gamma^\phi_{\psi\phi} = \cot\psi\,. \tag{43}$$

So the non-vanishing Christoffel symbols are

$$\begin{aligned}\Gamma^\psi_{\theta\theta} &= -\sin\psi\cos\psi\,, & \Gamma^\psi_{\phi\phi} &= -\sin\psi\cos\psi\sin^2\theta\,, \\ \Gamma^\theta_{\psi\theta} &= \cot\psi\,, & \Gamma^\theta_{\phi\phi} &= -\sin\theta\cos\theta\,, \\ \Gamma^\phi_{\theta\phi} &= \cot\theta\,, & \Gamma^\phi_{\psi\phi} &= \cot\psi\,.\end{aligned} \tag{44}$$

(b) The Riemann tensor is given in terms of Christoffel symbols as

$$R^{\rho}{}_{\sigma\mu\nu} = \partial_\mu \Gamma^{\rho}_{\nu\sigma} - \partial_\nu \Gamma^{\rho}_{\mu\sigma} + \Gamma^{\rho}_{\mu\lambda}\Gamma^{\lambda}_{\nu\sigma} - \Gamma^{\rho}_{\nu\lambda}\Gamma^{\lambda}_{\mu\sigma} \,. \tag{45}$$

The symmetries of the Riemann tensor tell us that

$$R^{\bar\mu}{}_{\bar\mu\bar\nu\bar\lambda} = 0 \,, \qquad R^{\bar\mu}{}_{\bar\lambda\bar\nu\bar\nu} = 0 \,, \tag{46}$$

where we are also using the fact that the metric we are studying is diagonal. Let us compute, for example,

$$\begin{aligned} R^{\theta}{}_{\psi\psi\theta} &= \partial_\psi \Gamma^{\theta}_{\theta\psi} - \partial_\theta \Gamma^{\theta}_{\psi\psi} + \Gamma^{\theta}_{\psi\lambda}\Gamma^{\lambda}_{\theta\psi} - \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\psi\psi} \\ &= \partial_\psi \Gamma^{\theta}_{\theta\psi} + \Gamma^{\theta}_{\psi\theta}\Gamma^{\theta}_{\theta\psi} \\ &= \partial_\psi \cot\psi + \cot^2\psi \\ &= -1 \end{aligned} \tag{47}$$

From this we can use the symmetries to deduce some other components

$$\begin{aligned} R^{\theta}{}_{\psi\theta\psi} = g^{\theta\theta} R_{\theta\psi\theta\psi} = -g^{\theta\theta} R_{\theta\psi\psi\theta} = -R^{\theta}{}_{\psi\psi\theta} = 1 \,, \\ R^{\psi}{}_{\theta\psi\theta} = g^{\psi\psi} R_{\psi\theta\psi\theta} = -g^{\psi\psi} R_{\theta\psi\psi\theta} = -g^{\psi\psi} g_{\theta\theta} R^{\theta}{}_{\psi\psi\theta} = \sin^2\psi \,, \\ R^{\psi}{}_{\theta\theta\psi} = -R^{\psi}{}_{\theta\psi\theta} = -\sin^2\psi \,. \end{aligned} \tag{48}$$

Then we compute

$$\begin{aligned} R^{\theta}{}_{\phi\phi\theta} &= \partial_\theta \sin\theta\cos\theta + \Gamma^{\theta}_{\phi\lambda}\Gamma^{\lambda}_{\theta\phi} - \Gamma^{\theta}_{\theta\lambda}\Gamma^{\lambda}_{\phi\phi} \\ &= \cos 2\theta + \Gamma^{\theta}_{\phi\phi}\Gamma^{\phi}_{\theta\phi} - \Gamma^{\theta}_{\theta\psi}\Gamma^{\psi}_{\phi\phi} \\ &= \cos 2\theta - \cos^2\theta + \cos^2\psi\sin^2\theta \\ &= -\sin^2\theta\sin^2\psi \end{aligned} \tag{49}$$

Analogously to what we did before, we find

$$\begin{aligned} R^{\theta}{}_{\phi\theta\phi} = -R^{\theta}{}_{\phi\phi\theta} = \sin^2\theta\sin^2\psi \,, \\ R^{\phi}{}_{\theta\phi\theta} = -g^{\phi\phi} g_{\theta\theta} R^{\theta}{}_{\phi\phi\theta} = \sin^2\psi \,, \\ R^{\phi}{}_{\theta\theta\phi} = -R^{\phi}{}_{\theta\phi\theta} = -\sin^2\psi \,. \end{aligned} \tag{50}$$

Finally we compute

$$\begin{aligned} R^{\psi}{}_{\phi\phi\psi} &= -\partial_\psi \Gamma^{\psi}_{\phi\phi} + \Gamma^{\psi}_{\phi\lambda}\Gamma^{\lambda}_{\phi\psi} \\ &= -\cos 2\psi \sin^2\theta + \Gamma^{\psi}_{\phi\phi}\Gamma^{\phi}_{\phi\psi} \\ &= -\cos 2\psi \sin^2\theta - \cos^2\psi\sin^2\theta \\ &= -\sin^2\psi\sin^2\theta \,, \end{aligned} \tag{51}$$

which we use to find

$$\begin{aligned} R^{\psi}{}_{\phi\psi\phi} &= \sin^2\psi\sin^2\phi \,, \\ R^{\phi}{}_{\psi\psi\phi} &= -1 \,, \\ R^{\phi}{}_{\psi\phi\psi} &= 1 \,. \end{aligned} \tag{52}$$

The Riemann tensor in three dimensions has $\frac{3^2}{12}(3^2-1) = 6$ independent components. We have computed three. The remaining ones are of the form $R^{\bar{\mu}}{}_{\bar{\nu}\bar{\mu}\bar{\lambda}}$, like

$$\begin{aligned} R^{\theta}{}_{\phi\theta\psi} &= 0 \\ R^{\phi}{}_{\theta\phi\psi} &= 0 \\ R^{\psi}{}_{\phi\psi\theta} &= 0\,. \end{aligned} \tag{53}$$

which all happen to vanish. We have found all the 6 independent components, and now we can move to the computation of the Ricci tensor. The components of the Ricci tensor are given by

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}\,. \tag{54}$$

By considering the list of non-vanishing Riemann tensor components we found, it is clear that only the diagonal components of the Ricci tensor will be non-zero. We find

$$\begin{aligned} R_{\psi\psi} &= R^{\theta}{}_{\psi\theta\psi} + R^{\phi}{}_{\psi\phi\psi} = 1 + 1 = 2\,, \\ R_{\theta\theta} &= R^{\psi}{}_{\theta\psi\theta} + R^{\phi}{}_{\theta\phi\theta} = \sin^2\psi + \sin^2\psi = 2\sin^2\psi\,, \\ R_{\phi\phi} &= R^{\psi}{}_{\phi\psi\phi} + R^{\theta}{}_{\phi\theta\phi} = \sin^2\psi\sin^2\phi + \sin^2\theta\sin^2\psi = 2\sin^2\psi\sin^2\phi\,. \end{aligned} \tag{55}$$

The Ricci scalar is given by

$$R = R^{\mu}{}_{\mu} = g^{\mu\nu}R_{\mu\nu} = g^{\psi\psi}R_{\psi\psi} + g^{\theta\theta}R_{\theta\theta} + g^{\phi\phi}R_{\phi\phi} = 2 + 2 + 2 = 6\,, \tag{56}$$

confirming the fact that S^3 has constant curvature. If we were to reintroduce the radius, we would find that

$$R = \frac{6}{r^2}\,. \tag{57}$$

The curvature of S^3 is inversely proportional to the radius of the sphere. Can you argue why the units of R should be $[L]^{-2}$?

(c) We want to check that

$$R_{\rho\sigma\mu\nu} = \frac{R}{n(n-1)}(g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}) = g_{\rho\mu}g_{\sigma\nu} - g_{\rho\nu}g_{\sigma\mu}\,. \tag{58}$$

We have found three non vanishing independent components of the Riemann tensor. We thus need to perform only three checks

$$\begin{aligned} R_{\psi\theta\theta\psi} &= g_{\psi\psi}R^{\psi}{}_{\theta\theta\psi} = -\sin^2\psi = g_{\psi\theta}g_{\theta\psi} - g_{\psi\psi}g_{\theta\theta} = -\sin^2\psi \\ R_{\psi\phi\phi\psi} &= g_{\psi\psi}R^{\psi}{}_{\phi\phi\psi} = -\sin^2\psi\sin^2\theta = g_{\psi\phi}g_{\phi\psi} - g_{\psi\psi}g_{\phi\phi} = -\sin^2\psi\sin^2\theta\,, \\ R_{\theta\phi\phi\theta} &= g_{\theta\theta}R^{\theta}{}_{\phi\phi\theta} = -\sin^4\psi\sin^2\theta = g_{\theta\phi}g_{\phi\theta} - g_{\theta\theta}g_{\phi\phi} = -\sin^4\psi\sin^2\theta\,, \end{aligned} \tag{59}$$

as we wanted to show.

(d) There are two indirect ways to argue that the Weyl tensor on this space has to vanish.

- The Weyl tensor always vanishes in three dimensions. This follows from the symmetry properties

  $$C_{\rho\sigma\mu\nu} = C_{[\rho\sigma][\mu\nu]}\,, \qquad C_{\rho\sigma\mu\nu} = C_{\mu\nu\rho\sigma}\,, \qquad C_{\rho[\sigma\mu\nu]}, \tag{60}$$

  together with the tracelessness condition and the fact that in three dimensions the indices can only take three different values.

- Two manifolds are said to be related by a Weyl transformation if they have metrics that can be written as

$$\tilde{g}_{\mu\nu} = \Omega^2(x) g_{\mu\nu}. \tag{61}$$

  If we map the sphere through a stereographic projection, its metric $\tilde{g}_{\mu\nu}$ is related to the metric of flat space with $\Omega(x) = \frac{1}{1+|x|^2}$. Metrics that are related by a Weyl transformation have the same Weyl tensor, up to a constant factor. Since the Weyl tensor of Euclidean space is identically vanishing because there is no curvature, the Weyl tensor on a sphere also has to be identically vanishing.